An approach to renormalization on the n-torus

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The coding theory of rotations (by inspecting closely their relation to flows) and the continued fractions algorithm (by considering even two-coloring of the integers with a given proportion of, say, blue and red) are revisited. Then, even n-coloring of the integers is defined. This allows one to code rotations on the $(n-1)$-torus by considering linear flows on the n-torus and yields a simple geometric approach to renormalization on tori by first return maps on the coding regions.

I. INTRODUCTION

"Universality," also called "rigidity" or "geometric rigidity,"[1] is certainly the most striking aspect of renormalization group theory applied to dynamical systems.[2,3] Universality occurs in dynamics when some topological types of recurrent orbits (the complete list of which is not yet known) exist in the presence of enough smoothness. [In a slightly more general context, rigidity means in fact that topologically (or, e.g., homotopically) identical objects are also geometrically similar under some structure hypothesis.] Even when restricting our studies to dynamical problems, it is not clear how many of the universality properties can be understood in the framework of renormalization. Renormalization might often be seen as a sophisticated way to speak about quasi-self-similarity, where the "quasiness" may change either according to context or to taste. Hence, universality occurs when metric (quasi) self-similarity is obtained upon measuring objects that present the same kind of topological self-similarity.

When dealing with dynamical systems, it is usually easier to formulate a (topological) renormalization scheme at the level of continuous maps or at the level of a typical family of maps with the topological dynamics of interest. This has some intrinsic applications (e.g., the description of the boundary of positive topological entropy) and is a necessary step in understanding the role of the smooth structure, i.e., before addressing the universality problem *per se*. The types of recurrence that are now understood in this context are rotations on the circle[4–6] or the p-adics,[1–3,6] as well as generalizations of the latter in the form of automorphisms on trees.[1,7] Thue–Morse sequences and some of their generalizations have also been considered.[6] Here, we sketch our first steps toward developing a renormalization group theory for rotations on tori of arbitrary dimension. Although our arguments are mathematically elementary, they involve graphical and combinatorial constructions, which the interested reader is urged to work out for him(her)self.

II. LINEAR FLOWS IN R^2 AND FOUR ROTATIONS THEY SUSPEND

Let F_α be an oriented foliation of the real plane $\mathbf{R}^2$ by straight lines (i.e., a decomposition of $\mathbf{R}^2$ into oriented parallel straight lines called *leaves* in this context) that are chosen as neither horizontal nor vertical to simplify the discussion. Such an F_α is determined by the angle that gives its direction and is induced by the flow of the constant vector field with vector $\mathbf{V}_\alpha=(1,\alpha)$, $\alpha>0$. Here, F_α and $\mathbf{V}_\alpha$ can be considered as lifts to $\mathbf{R}^2$ of a foliation $\mathscr{F}_\alpha$ and a vector field $\mathscr{V}_\alpha$ on the two-torus $\mathbf{T}^2=\mathbf{R}^2/\mathbf{Z}^2$. Lifts of the simplest sections of the torus are furnished by the families of horizontal unit intervals,

$$I_{h,m,n}=\{(x,y)\,|\,m\leqslant x<m+1,y=n\},$$

and vertical unit intervals,

$$I_{v,m,n}=\{(x,y)\,|\,x=m,n\leqslant y<n+1\}.$$

Let $[x]$ stand for the integer part of x, and $\{x\}$ for its fractional part. It is immediate that the next horizontal unit interval crossed by the flow line starting at $(m+x,n)\in I_{h,m,n}$, $0\leqslant x<1$, is $I_{h,m+[1/\alpha+x],n+1}$, and that the crossing will take place at the point $(m+1/\alpha+x,n+1)$. This means that the flow induced by $\mathscr{V}_\alpha$ is a suspension of the rotation $R_{\{1/\alpha\}}$ on the standard circle $\mathbf{T}=\mathbf{R}/\mathbf{Z}$ with length 1. For later use, we remark that the closed leaf segment in between two consecutive horizontal crossing contains either $[1/\alpha]$ or $[1/\alpha]+1$ vertical crossings. Similarly, consideration of the vertical segments yields that the flow induced by $\mathscr{V}_\alpha$ is simultaneously a suspension of the rotation $R_{\{\alpha\}}$.

In order to obtain another pair of flows, consider the lattice of horizontal and vertical lines joining all points of

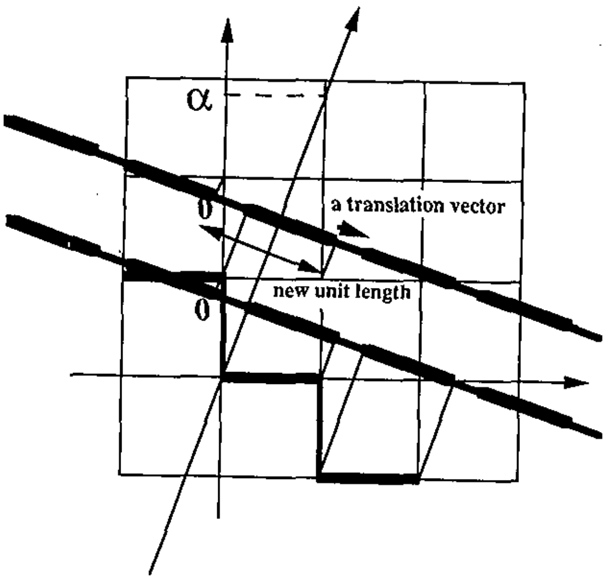

FIG. 1. Constructions for the third and fourth maps suspended by a linear flow.

$\mathbf{Z}^2$ as a union of descending "staircases" with generic step composed of the union of $I_{h,m,n}$, $I_{v,m+1,n-1}$ and the joining point $(m+1,n)$. Fix an "origin" on each of these infinite staircases in such a way that all origins are aligned, parallel to the vertical axis, and at a distance one apart of each other. (Note that in the case in which the origins lie on the horizontal part of the step, the second condition is automatic.) Now project each staircase with its marked point along the foliation F_α onto a straight line transverse to F_α, and choose the length of the image of a single step as the unit length of the image marked line (see Fig. 1). Each horizontal segment of a step projects to a segment of length $\alpha/(1+\alpha)$, while vertical segments have projections of length $1/(1+\alpha)$. The marked straight lines are ordered along the flow lines, and their orientation is chosen so that moving to the right increases the coordinate. If each marked straight line is identified with the next one so that the marked points are matched together, then the flow induced by $\mathscr{V}_\alpha$ is furthermore a suspension of the rotation $R_{\{1/(1+\alpha)\}}$. On the other hand, the flow induced by $\mathscr{V}_\alpha$ may also be viewed as a suspension of the rotation $R_{\{\alpha/(1+\alpha)\}}$ by choosing the origins on the staircases so that they align horizontally, and by reversing the orientation on the image straight lines.

At this point, the stage is set to present a way to think about renormalization of rotations on the circle and generalizations to higher-dimensional tori. However, we first invite the reader to a detour, mainly because there is a well-developed theory of coding for rotations on the circle,[8–14] which allows an alternate approach to renormalization[6] as sketched in Sec. V. So we shall pause a moment to describe (Secs. III and IV) this coding theory which has its own interest and which we shall also generalize to higher dimension.

III. EVEN TWO-COLORINGS OF THE INTEGERS AND CONTINUED FRACTIONS

For the sake of brevity, we shall consider here only rational ratios: As usual one can compute formulas from combinatorics in the rational case and use them afterward in the real case by continuity (see, e.g., Ref. 6). Thus let a and b be two positive coprime integers. Consider the problem of coloring "as evenly as possible" the lattice of integers with colors blue and red so that in the mean, there are "a" blue points for every "b" red ones. To make the problem even more concrete, try to place evenly "a" apples and "b" oranges on a circle. There is an iterative way to achieve such a distribution, by successive approximations, which also will serve as a precise definition of optimal evenness. Furthermore, in doing so, the algorithm will give successive approximations to the proportions of each fruit which are the convergents of the respective continued fractions. It goes as follows, assuming that $a<b$:

Step 1: 1-1: "grouping." We have two groups of objects: (i) a first group made of b blocks of one orange, and (ii) a second group made of a blocks of one apple.

1-2: "approximation." Consider the proportions in any block of the first group, and get: (i) a proportion of 0/1 apples and (ii) a proportion of 1/1 oranges.

Step 2: 2-1: "grouping." We form two groups of objects: (i) a first group made of a blocks, each composed of one apple, followed to the right by as many oranges as can be distributed evenly to a apples, i.e., $[b/a]$ oranges, and (ii) a second residual group made of the residue $b \bmod a (=b_a)$ blocks of one orange.

2-2: "approximation." Consider the proportions of each fruit in any given block of the first group and get: (i) a proportion of $1/(1+[b/a])$ apples and (ii) a proportion of $[b/a]/(1+[b/a])$ oranges. If

$$\frac{1}{1+[b/a]}=\frac{a}{a+b},$$

then

$$\frac{[b/a]}{1+[b/a]}=\frac{b}{a+b}.$$

Also, in this case we are done and we can place our apples and oranges on the circle, having learned nothing. So, let us assume that $a\neq 1$, and describe the following generic step.

Step 3: 3-1: "grouping." We form two groups of objects: (i) a first group made of b_a blocks, each composed of one orange, followed to the right by as many blocks of one apple and $[b/a]$ oranges (i.e., blocks from the previous first group) as one can distribute evenly to b_a oranges, and (ii) a second group made of the remaining a_{b_a} blocks of one apple followed by $[b/a]$ oranges.

3-2 "approximation." Consider the proportions in any block of the second group, and get: (i) a proportion

$$\frac{[a/b_a]}{1+[a/b_a]+[a/b_a]\cdot[b/a]}$$

of apples, and (ii) a proportion

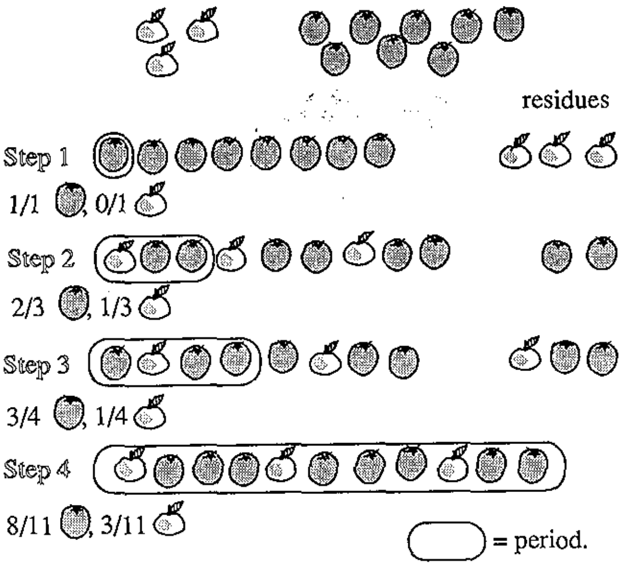

FIG. 2. Construction of an even distribution of 3/11 of apples and 8/11 of oranges on the line.

$$\frac{1+[a/b_a]\cdot[b/a]}{1+[a/b_a]+[a/b_a]\cdot[b/a]}$$

of oranges.

The process continues in this fashion. Note that at each stage the second "residual" group will be composed of fewer blocks than the first group. If the residual group is composed of more than one block, then distribute the blocks of the first group over those of the second to form a new first group and new residue. If at some step the second group is a single block, then the next step is the last one and consists in putting the entire first group to the right of the second. As a small picture is often better than an obscure description, we illustrate this process in Fig. 2.

There are two main remarks to be made about the algorithm described above. The first is about raw numbers: The above algorithm amounts to computing simultaneously the continued fraction expansions of $a/(a+b)$, $b/(a+b)$, and (necessarily) a/b. As we mentioned earlier the general continued fraction algorithm follows by continuity. The second remark explains the connection to dynamics and is postponed to the next section.

In Sec. VI, we shall reinterpret the two-colorings just constructed in a way that allows us to propose a general definition of even n-coloring for any $n>0$. We recently came across a paper by Smith[8] examining a particular case of the even two-colorings as defined in Sec. VI. For an alternative approach to even two-coloring, we refer to Ref. 9.

IV. BACK TO ROTATIONS

We shall recall here some simple definitions and facts about the symbolic dynamics of rotations. An important piece of this theory can be formulated as a renormalization analysis that will be presented in Sec. V.

The symbolic dynamics of rotations goes back to Hedlund and Morse[10] (see also Ref. 11 for complementary information). It is remarkable that equivalent arithmetic problems were solved by Jean Bernoulli[12] and Markoff,[13] and independently by Christoffel,[14] and again in the aforementioned paper by Smith.[8]

When considering the rotation R_β with angle β, the circle is split into two half-open segments, a blue one of length β, and a red one of length $1-\beta$. Define the *address* of a point as 0 if it belongs to the red part, and 1 if it belongs to the blue part. Since rotations are invertible maps, it is convenient to define the orbit of a point x as the set $\{R_\beta{}^n(x)\}_{n\in\mathbf{Z}}$. Then the *itinerary* of a point is the ordered sequence of addresses of the points $\{R_\beta{}^n(x)\}_{n\in\mathbf{Z}}$.

Equivalently, if $\beta=1/(1+\alpha)$, consider the third type of suspension described in Sec. II. Follow both forward and backward the flow line through a point of coordinate x in any tilted line, and write down a 0 for any crossing with a horizontal segment, a 1 for any crossing with a vertical segment. The second remark related to apples and oranges is that the sequence of 0's and 1's corresponding to itineraries of rotations are distributed as evenly as possible. Notice that in the case of a rational angle, there is a single symbolic orbit, up to a shift. In the case of a rotation with an irrational angle, there is a single closure for the symbolic orbit in the shift space.

V. RENORMALIZATION

Any symbolic orbit of a rotation can be written with at most two types of blocks of symbols in many different ways. For example, we may rewrite it as a sequence of blocks of the form 01^n and 01^{n+1}, where n depends only on the rotation number β. If $\beta\leqslant\frac{1}{2}$, then $n=0$.

To describe renormalization in a way that uses the symbolic theory, split a sequence into blocks by cutting it before each 0. To form a second symbolic sequence now replace the blocks 01^n by 0 and the blocks 01^{n+1} by 1. If $\beta=1/(1+\alpha)$ then the new sequence is in fact a symbolic sequence for the rotation with angle $\{1/\alpha\}$. By definition this rotation $R_{\{1/\alpha\}}$ is obtained by renormalization from R_β, since it can be obtained from R_β by taking an induced map. More precisely, in this case the inducing region (i.e., the place where we take the first return map) is the red segment. If the original rotation R_β corresponds to the third suspension constructed in Sec. II, then the renormalized rotation $R_{\{1/\alpha\}}$ is simply the first suspension.

By using a different grouping and the symbols and renaming scheme, the code for the rotation $R_{\{\alpha\}}$ can be obtained (an exercise we leave to the reader). This rotation can also be obtained from R_β by renormalization, the induction being done this time on the blue segment. Thus, reconsidering what was done in Sec. II, one can say that the same flow suspends two rotations together with two of their renormalized maps.

Note that a further block decomposition of the symbolic sequences would also yield the more common transformation $x\rightarrow\{1/x\}$ on rotation numbers, after proper renaming of the blocks by 0 and 1. This is precisely the

Gauss map, i.e., the truncation of the continued fraction that was used in the original papers on renormalization for circle maps.[4,5] For generalities on the relationship among symbols, induction, and renormalization, see Ref. 6.

VI. EVEN n-COLORING

For $n=2$, an even two-coloring of $\mathbf{Z}$ has been defined in Sec. III, in fact merely by the method proposed to generate it. Let us then remark that the way to distribute evenly a apples and b oranges around a circle when worrying only about their mutual order, can also be described as follows: place all apples at a distance $1/a$ apart on one circle; place all oranges at a distance $1/b$ apart on a second circle of same size; superimpose the two circles in such a way that no apple is in the same place as an orange, forming an apple and orange circle.

For $n=3$, continue the algorithm as follows. If you have c bananas, place them around a third same circle at a distance $1/c$ apart. Next superimpose the banana circle to the apple and orange circle, in such a way that no two fruits are exactly at the same place. What you have generated is by definition an even three-coloring, and the way to generalize to n-coloring is now obvious. It is also an easy property to show that for an even n-coloring with $n>2$, any $(n-1)$-coloring obtained from it by erasing all points of any single color is also even. Similarly, all two-colorings induced from an even n-coloring by erasing $n-2$ colors are even. Unfortunately, these are NOT characteristic properties.

A main difficulty is that there are many even n-colorings for $n>2$. In the case of irrational proportions, you proceed the same way. Now, however, you have to do it on the universal cover: place infinitely many fruits on the real line with a spacing inverse to the proportion of that fruit. An example of a complete list of even three-colorings with the same proportions is given in Fig. 3. We have chosen to use straight lines instead of circles (i.e., the universal cover version) to illustrate our rational example in Fig. 3.

In the case of n irrational proportions with no mutual rational relations, it is in fact a consequence of our theory that any two even n-colorings with the same proportions have shifts that are arbitrarily close to each other.

VII. CODES AND RENORMALIZATION FOR ROTATIONS ON THE n-TORUS

In order to code rotations on $\mathbf{T}^n$, we need to split the torus into regions that will define addresses, generating a coding so that the rotation vector can be recognized as in the case of the circle. We obtain such splittings by using the most obvious generalization of the construction reported in Sec. II. More precisely, we start from a linear flow in $\mathbf{R}^{n+1}$, and split the lattice of hyperplanes with integer coordinates into staircases. By projecting along the flow direction on a transverse hyperplane, assuming that all coordinates of the vectors of the vector field are positive, we then get tilings of $\mathbf{R}^n$ which are characterized by the volumes of the tiles. All tiles are semiopen parallelepipeds, i.e., affine copies of $\{x_1 \cdots x_n \mid 0 \leqslant x_i < 1 \text{ for } 1 \leqslant i \leqslant n\}$. An

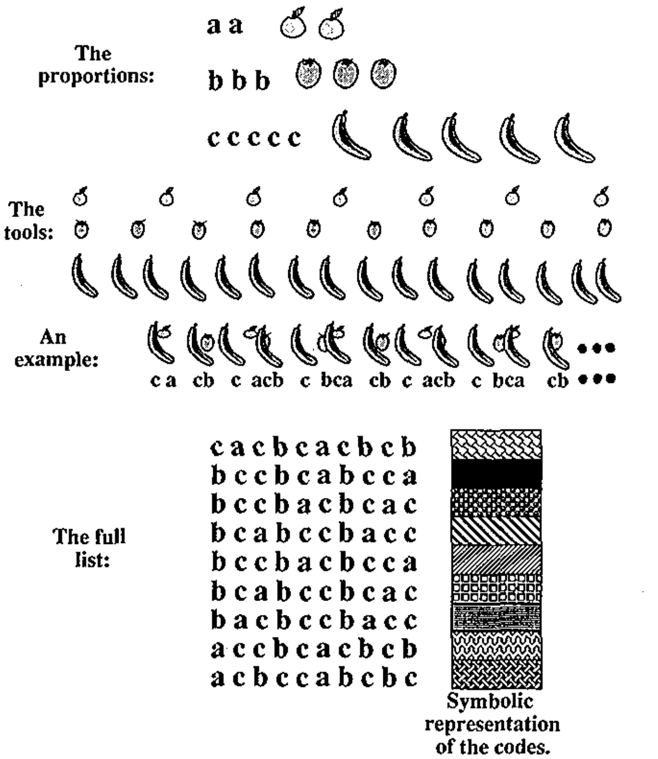

FIG. 3. Construction of an even distribution of 2/10 apples, 3/10 oranges, and 5/10 bananas on the line.

example is given in Fig. 4. This gives tilings of n-dimensional tori by splitting $\mathbf{R}^n$ in parallelepipeds of unit volume, which correspond, up to orientation, to elements of $\mathrm{Gl}(n,\mathbf{Z})$.

We have the following result by merely formalizing the discussion presented above.

Theorem: Let $A=(a_1,\ldots,a_n)$, with $S=\sum_{i=1}^{n}|a_i| \leqslant 1$ and $0 \leqslant |a_i| \leqslant 1$ be a vector in $\mathbf{R}^n$, M be an element of $\mathrm{Gl}(n,\mathbf{Z})$, L_M be the image of $\mathbf{Z}^n$ in $\mathbf{R}^n$ under M, $\mathbf{T}_M^n$ be

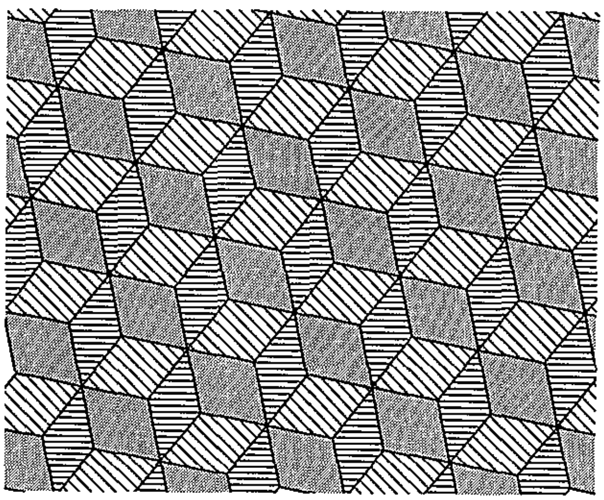

FIG. 4. Typical tiling achieved by projecting a staircase in $\mathbf{R}^3$ on a plane.

$\mathbf{R}^n/L_M$ with π_M the associated projection, and $R_{A,M}$ be the rotation on $\mathbf{T}^n_M$ whose lift to $\mathbf{R}^n$ by π_M is $x \to x + A$.

Assume A is orthogonal to m axes of $\mathbf{R}^n$, where $0 \leqslant m \leqslant n$. Then there exists a splitting of $\mathbf{T}^n_M$ into $p = n+1-m$ semiopen parallelepipeds P_k such that: (1) after proper identification of pairs of faces, the first return map in each P_k is a rotation of $\mathbf{T}^n$. (2) The P_k's serve as coding regions for $R_{A,M}$ so that any code for a point in a P_k is an even n-coloring with proportions

$$|a_1|,...,|a_n|,\ 1 - \sum_{i=1}^{n} |a_i|.$$

A main consequence of (1) above is that rescaling the first return map yields a renormalized map in a sense that generalizes what is usually done (sometimes with a different point of view) in the cases of period doubling and circle maps.[6] A previous approach to renormalization on two-tori was announced in Ref. 15. One advantage of our point of view is that it does not require any preliminary knowledge of simultaneous Diophantine approximation, a general theory much less complete than Diophantine approximation of a single irrational number. This arithmetic deep difficulty does limit applications of renormalization methods whose formulation is based on number theory.

The condition $S \leqslant 1$ in the Theorem is irrelevant in dimensions $n=1$ and $n=2$, since for instance any a_i with $a_i > 1/2$ can be replaced by $1 - a_i$ to represent the same rotation. It is shown in Ref. 16 that in higher dimensions, if $S > 1$ for the shortest vector representing the rotation, one can either iterate the map until the vector satisfies $S \leqslant 1$, or give up the splitting of the torus into renormalization regions and be content with the fact that one region at least can always be chosen so that its first return map is a rotation. The coding when $S > 1$ is also described in Ref. 16.

The symbolic approach to renormalization in dynamical systems introduced in Ref. 6 now extends to arbitrary dimension, although we do not know how to generalize the block structure of sequences described in Sec. V. However, the codes do allow us to write automatically the formulas for the renormalized maps. As an example, consider a code beginning *abbca*..., and denote by F_x the restriction of the rotation to the coding region x. Then the return map to the region a, in the domain where the code begins as above, reads $F_c o F_b o F_b o F_a$. It is easy to show that there are only finitely many different such formulas for each rotation in any dimension.

Figure 5 shows the main geometric constructions allowing us to construct the coding regions when M is the unit matrix, with the same $a=2$, $b=3$, $c=5$ used previously in Fig. 3 to illustrate the construction of even colorings. There the vector **A** is one of the rotation vectors, the other ones being **B** and **C**. The vector **D** joins a point to its closest neighbor on its own orbit. Figure 5 also shows how every fundamental region of the corresponding rotation on the torus splits according to the various codes defined up to a shift. The three coding regions in Fig. 5 are, respectively, homotopic to the elements (1,0), (0,1), and (1,1) of $\pi_1(\mathbf{T}^2)$. Regions corresponding to the triple $[(m,n),(p,q),(m+p,n+q)]$ can be realized if and only if the matrix $\begin{pmatrix} m & p \\ n & q \end{pmatrix}$ is in $\mathrm{Gl}(2,\mathbf{Z})$. Examples are given in Fig. 6. Each such matrix yields a different relationship between the codes and the rotation vectors. For rotations in dimension $n > 2$, $\mathrm{Gl}(n,\mathbf{Z})$ similarly classifies the splittings in $(n+1)$ coding regions.

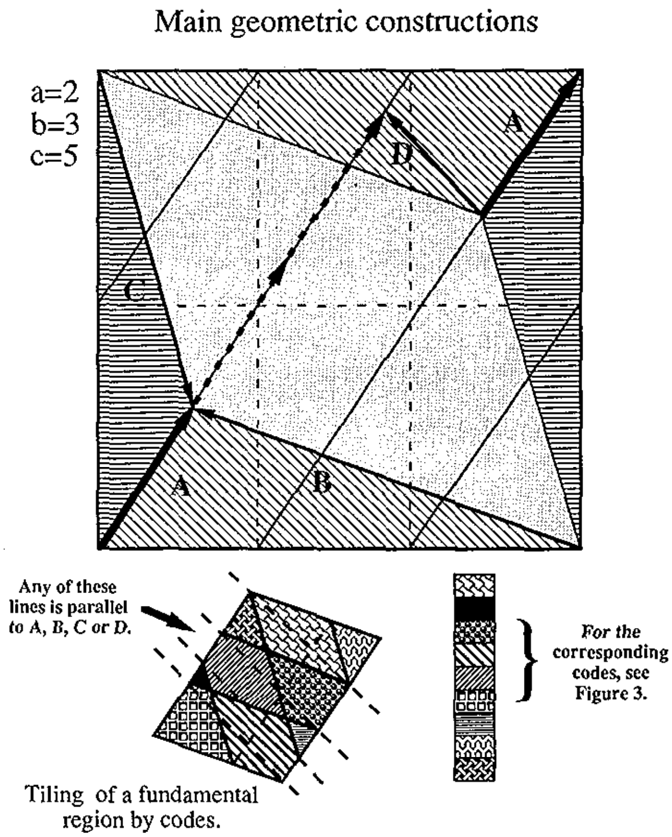

FIG. 5. Construction of the coding regions when M is the unit matrix. A tiling of a fundamental region by codes is also shown: This tiling only makes sense in the case of a vector with rational coordinates.

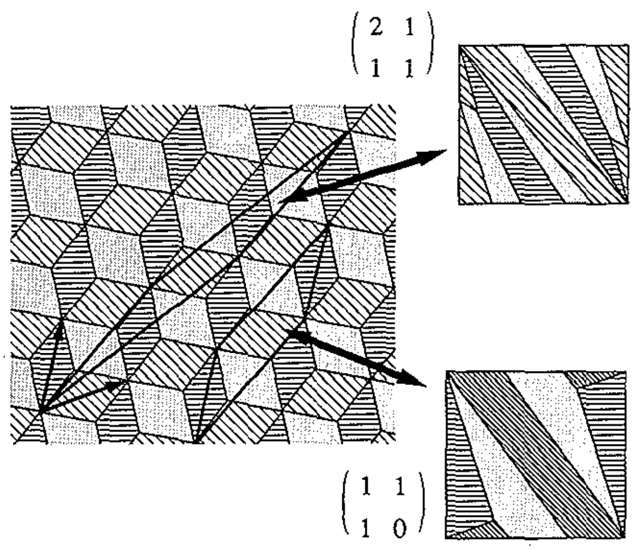

FIG. 6. Examples of coding regions for the case of two nontrivial matrices.

There are many consequences to these simple considerations being worked out: mainly relations with the theory of simultaneous Diophantine approximations, coding and renormalization of homeomorphisms and diffeomorphisms including Denjoy theory[17] in arbitrary dimension. Precise (in the sense of mathematical) statements and proofs, including details of the material presented here, will appear in Ref. 16.

A priori quite different coding methods and tilings of the plane have been developed by Rauzy[18] and independently by Thurston;[19] these results also have a flavor of renormalization group theory.

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